Problem 3

I. Answer the following questions on information theory. Let $S$ be a Markov information source that has three states $\{s_0, s_1, s_2\}$. For each trial, $S$ makes a transition to the next state according to the transition probability matrix $T$ given by the following equation.

$$
T = \begin{bmatrix}
p(s_0|s_0) & p(s_1|s_0) & p(s_2|s_0) \\
p(s_0|s_1) & p(s_1|s_1) & p(s_2|s_1) \\
p(s_0|s_2) & p(s_1|s_2) & p(s_2|s_2)
\end{bmatrix} = \begin{bmatrix}
0.3 & 0.7 & 0.0 \\
0.3 & 0.5 & 0.2 \\
0.8 & 0 & 0.2
\end{bmatrix}
$$

During each transition, $S$ outputs the next state. You may use the following approximations when necessary: $\log_2 3 \approx 1.58$, $\log_2 5 \approx 2.32$, and $\log_2 7 \approx 2.81$.

(1) Draw the state transition diagram of $S$.

(2) Obtain the probabilities $P_0$, $P_1$, and $P_2$ of being in the states $s_0$, $s_1$, and $s_2$, respectively, after a sufficient number of trials.

(3) Obtain the overall entropy of $S$. Express the answer using $P_0$, $P_1$, and $P_2$ defined in Question (2).

(4) Consider encoding two successive outputs of $S$ into code words.

(4-i) Design a set of code words that maximizes the encoding efficiency.

(4-ii) Describe how to quantify the encoding efficiency of the set of code words designed in Question (4-i).
II. Answer the following questions on signal processing. The Fourier transform $F(\omega)$ of a continuous-time signal $f(t)$ is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt,$$

where $t (-\infty < t < \infty)$ is the time, $\omega$ is the angular frequency, and $j$ is the imaginary unit. Its inverse Fourier transform is given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega.$$

The unit impulse function $\delta(t)$ is expressed as

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \text{ and } \int_{-\infty}^{\infty} \delta(t) dt = 1.$$ 

The complex Fourier series of a periodic function $g(t)$ with a period $T$ is expressed as

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T} nt}, \text{ where } c_n = \frac{1}{T} \int_{\frac{T}{2}}^{\frac{T}{2}} g(t) e^{-j\frac{2\pi}{T} nt} dt \text{ (n: integer)}.$$ 

(1) Let $X(\omega)$ be the Fourier transform of a real continuous-time signal $x(t)$. Show that its complex conjugate $X^*(\omega)$ is equal to $X(-\omega)$.  

(2) Obtain the Fourier transform of the unit impulse function.  

(3) Assume that a signal $x(t)$ is input to a low-pass filter and the output signal $y(t)$ is obtained, and that the low-pass filter has an ideal frequency response $H(\omega)$ defined by

$$H(\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \text{otherwise} \end{cases},$$

where $\omega_c (>0)$ is the angular cutoff frequency. Obtain the output signal $y(t)$ assuming that $x(t)$ is the unit impulse. Also sketch an outline of $y(t)$, and indicate the value of $y(0)$ and value(s) of $t$ where $y(t) = 0$.  

(4) A periodic impulse train $d(t)$ with a period of $T (>0)$ is expressed as

$$d(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

(4-i) Express $d(t)$ in the form of the complex Fourier series.  

(4-ii) Show that $D(\omega)$ consists of a periodic impulse train, where $D(\omega)$ is the Fourier transform of $d(t)$.  

(5) Consider the sampling of a real continuous-time signal $x(t)$.  

(5-i) Explain how to recover $x(t)$ from the sampled signals in a few lines. You may use Eqs. (iv) to (vi) if necessary.  

(5-ii) Describe the condition to recover $x(t)$ from the sampled signals.